

"Inverse Newton" = Newton's method for computing inverse functions.

$$f(g(x)) = x, \quad x \text{ in domain of } g.$$

To compute $y := g(x)$ solve

$$F(y) := f(y) - x = 0$$

with Newton's method:

$$y_0 \doteq g(x) \text{ initial guess}$$

$$\begin{aligned} y_{k+1} &= y_k - \frac{F(y_k)}{F'(y_k)} \\ &= y_k - \frac{f(y_k) - x}{f'(y_k)} \\ &= y_k + \frac{x - f(y_k)}{f'(y_k)} \end{aligned}$$

Main example.

$$f(x) = \ln x, \quad f'(x) = \frac{1}{x}, \quad x > 0$$

$$y = g(x) = e^x, \quad x \text{ real}$$

We may assume $x > 0$ because we know $e^0 = 1$ and $e^{-x} = \frac{1}{e^x}$.

$$y_0 := y_1 = e^x$$

$$y_{k+1} = y_k (1 + x - \ln y_k),$$

$$k = 0, 1, 2, \dots,$$

at least as long as $y_k > 0$!

Now $x > 0 \Rightarrow e^x > e^0 = 1$. Let's

start with $y_0 := 1$. Then

$y_1 = 1 + x > 1 = y_0$, so at least

$y_1 > 0$! We claim that $x > 0 \Rightarrow$

$$1 = y_0 < y_1 < \dots < y_k \nearrow y = e^x,$$

and quadratically fast, as we

shall see. We use (only!) the

mean value theorem.

□ Let us call

$$e_k := y - y_k = e^x - y_k$$

the kth error. Thus we want

to show that

$$e^x - 1 = e_0 > e_1 > \dots > e_k \searrow 0$$

quadratically fast. Now we have

$$\begin{aligned}
 y_{k+1} &= y_k + y_k (x - \ln y_k) \\
 &= y_k + y_k (\ln y - \ln y_k) \\
 &= y_k + y_k \underbrace{\frac{\ln y - \ln y_k}{y - y_k}}_{\substack{\text{---} \\ \leftarrow = f'(c_k) = \frac{1}{c_k}, c_k}} e_k
 \end{aligned}$$

(strictly) between y_k and y .

Assume inductively that $y_k < y$.

It's true for $k=0$! Then

$y_k < c_k < y$, so we can write

$$c_k = y_k + \theta_k (y - y_k) = y_k + \theta_k e_k$$

with $0 < \theta_k < 1$. Hence

$$\begin{aligned}
 e_{k+1} &= e_k - \frac{y_k}{y_k + \theta_k e_k} e_k \\
 &= \frac{\theta_k}{y_k + \theta_k e_k} e_k^2.
 \end{aligned}$$

Since $e_k = y - y_k > 0$ and

$0 < \theta_k < 1$ then

$$0 < \frac{\theta_k}{y_k + \theta_k e_k} < \frac{1}{y_k}$$

and so

$$0 < e_{k+1} < \frac{e_k^2}{y_k} \quad (*)$$

Hence

$1 = y_0 < y_1 < \dots < y_k \rightarrow y^* \leq y = e^x$,
and all the y_k are > 0 ! We
now let $k \rightarrow +\infty$ in the iteration
equation

$$y_{k+1} = y_k (1 + x - \ln y_k)$$

to get

$$y^* = y^* (1 + x - \ln y^*),$$

since the right side is a
continuous function of y_k for
 $y_k > 0$! That is, cancelling
 $y^* > 0$,

$$1 = 1 + x - \ln y^*,$$

that is

$$\ln y^* = x = \ln y.$$

Exponentiation gives

$$y^* = e^x = g(x).$$

Again, from (*) follows

$$0 < \frac{e_{k+1}}{e_k^2} < \frac{1}{y_k} \rightarrow \frac{1}{e^x}$$

showing at least quadratic convergence. In fact the relative error

$$r_k := \frac{e_k}{e^x} = \frac{e^x - y_k}{e^x}$$

satisfies

$$0 < \frac{r_{k+1}}{r_k^2} = e^x \frac{e_{k+1}}{e_k^2} < \frac{1}{y_k} < 1.$$

In general, when solving $f(x) = 0$ by Newton's method,

$$0 = f(x_k) + f'(x_k)(x_{k+1} - x_k),$$

we can use the next term of the Taylor development,

$$\begin{aligned} 0 = f(x^*) &= f(x_k + (x^* - x_k)) \\ &= f(x_k) + f'(x_k)(x^* - x_k) + \\ &\quad + \frac{1}{2} f''(\xi_k)(x^* - x_k)^2. \end{aligned}$$

Subtraction gives

$$0 = f'(x_k)(x^* - x_{k+1}) + \frac{1}{2} f''(\xi_k)(x^* - x_k)^2,$$

so

$$\frac{x_{k+1} - x^*}{(x_k - x^*)^2} = \frac{1}{2} \frac{f''(\xi_k)}{f'(x_k)} \rightarrow \frac{1}{2} \frac{f''(x^*)}{f'(x^*)},$$

provided $f'(x^*) \neq 0$. This is only a local convergence result.

In our case

$$0 < \frac{e_{k+1}}{e_k^2} \rightarrow -\frac{1}{2} \frac{F''(y)}{F'(y)} = \frac{1}{2e^x}$$

so

$$0 < \frac{r_{k+1}}{r_k^2} \rightarrow \frac{1}{2}. \quad \blacksquare$$

The next main example.

$$f(x) = a \tan x, \quad f'(x) = \frac{1}{1+x^2}$$

$$y = g(x) = \tan x, \quad |x| < \frac{\pi}{2}.$$

$\tan x$ odd and π -periodic

with poles at $x = \pm \frac{\pi}{2} \Rightarrow$

can assume $0 < x < \frac{\pi}{2}$.

The iteration is

$$y_0 := y = \tan x$$

$$y_{k+1} = y_k + (1 + y_k^2)(x - a \tan y_k),$$

$$k = 0, 1, 2, \dots$$

□ If we took $y_0 = 0$ we'd get $y_1 = x$, so let's use the "educated" initial guess $y_0 := x > 0$. Then $y_1 = x + (1 + x^2)(x - a \tan x)$. Is $y_1 > y_0$?

Same as: is $x > a \tan x$? Same as: is $\tan x > x$, since $\tan x$, and $a \tan x$, are increasing.

Same as $\frac{\sin x}{x} > \cos x$? This is true for $0 < x < \frac{\pi}{2}$ by the

"fundamental inequality of trig": $\cos x < \frac{\sin x}{x} < \frac{1}{\cos x}$,

for $0 < x < \frac{\pi}{2}$ (note all functions are even here, but equality holds when $x = 0$).

So we seem to be "going up".

Let's again define the

errors

$$e_k := y - y_k = \tan x - y_k.$$

Then, by the mean value theorem,

$$\begin{aligned} y_{k+1} &= y_k + (1 + y_k^2) \underbrace{\frac{\tan y - \tan y_k}{y - y_k}}_{= \frac{1}{1 + c_k^2}, c_k \text{ between } y_k \text{ and } y} e_k \\ &= y_k + \frac{1 + y_k^2}{1 + c_k^2} e_k, \end{aligned}$$

and

$$e_{k+1} = \left(1 - \frac{1 + y_k^2}{1 + c_k^2} \right) e_k.$$

Assume, inductively, that $0 < y_k < y$, that is $y_k > 0$ and $e_k > 0$. The first equality shows that $y_{k+1} > y_k$. Since c_k is strictly between y_k and y then $c_k = y_k + \theta_k y$ with $0 < \theta_k < 1$. So

$$\begin{aligned}
 e_{k+1} &= \frac{c_k^2 - y_k^2}{1 + c_k^2} e_k \\
 &= \theta_k \underbrace{\frac{2y_k + \theta_k e_k}{1 + (y_k + \theta_k e_k)^2}}_{> 0} e_k^2
 \end{aligned}$$

So $y_k < y_{k+1} < y$, and $y_k \rightarrow y^* \leq y$.
 By letting $y \rightarrow y^*$ in the main iteration we get at any $y^* = x$,
 that is $y^* = \tan x = y$. Finally

$$\begin{aligned}
 \frac{e_{k+1}}{e_k^2} &= \theta_k \frac{2y_k + \theta_k e_k}{1 + (y_k + \theta_k e_k)^2} \\
 &\rightarrow \frac{2}{y + 1/y} = \frac{2}{\cos x \sin x}
 \end{aligned}$$

and, because $0 < \theta_k < 1$,

$$\overline{\lim} \frac{e_{k+1}}{e_k^2} \leq \frac{2}{\cos x \sin x},$$

showing at least quadratic convergence. \blacksquare

What is \lim $\frac{e_{k+1}}{e_k^2}$?